C_o = total ionic concentration in the solution phase, meq./ml.
K_{AB} = selectivity coefficient for exchange of ions A and B
q = ionic concentration in the resin phase, meq./g. (dry)
Q_o = total ionic capacity of the resin phase, meq./g.
x = equivalent fraction of an ion in solution phase
X = equivalent fraction of an ion in solution phase based on only two ions
y = equivalent fraction of an ion in resin phase
Y = equivalent fraction of an ion in resin phase based on only two ions

Subscripts

R = resin phase
S = solution phase

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The Laminar-Turbulent Transition for Flow in Pipes, Concentric Annuli, and Parallel Plates

RICHARD W. HANKS
Union Carbide Nuclear Company, Oak Ridge, Tennessee

Recently Ryan and Johnson (*1*) proposed a stability parameter for pipe flow. For Newtonian fluids they showed that this parameter Z is proportional to the critical Reynolds number ($D\bar{v}\rho/\mu$) and is given by the relation

$$Z = 2\sqrt{\frac{1}{27}} N_{Re_c} \qquad (1)$$

They also demonstrated the utility of Z as a stability parameter for the isothermal pipe flow of power-law non-Newtonian fluids.

Hanks and Christiansen (*2*) extended the range of applicability of Ryan and Johnson's parameter to include heated flow of similar fluids, showing the temperature invariance of Z. However one may readily show the geometry dependence of Z by attempting to calculate the critical Reynolds number as observed by Davies and White (*3*) for flow between parallel plates. In order to calculate the experimental value one must postulate that Z be different for the two geometries.

In the present paper a generalized stability parameter will be proposed which is independent of the geometry of the flow system, is proportional to the Reynolds number for Newtonian flow, and contains Ryan and Johnson's (*1*) results for pipe flows in general as a special case.

DEVELOPMENT OF THE PARAMETER

In attempting to formulate a stability criterion of general applicability one should keep in mind the physical nature of such a parameter. It should be proportional to the conventional Reynolds number (which is a characteristic stability parameter) for the special case of Newtonian flow. The Reynolds number may be interpreted (*4*) physically as the ratio of the magnitude of certain inertial forces to the magnitude of the viscous forces acting on a fluid element. Therefore one might expect the generalized parameter to involve a similar ratio of magnitudes of forces.

The equations which describe the motion of a fluid are (*4*) the equation of continuity

$$\operatorname{div}(\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t} \qquad (2)$$

and the equations of motion

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \underset{(a)}{\frac{1}{2}\rho\, \mathbf{grad}(\mathbf{v}\cdot\mathbf{v})} - \underset{(b)}{\rho \mathbf{v} \times \zeta} = \underset{(c)}{\mathbf{F} - \mathbf{grad}\, p} - \underset{(d)}{\operatorname{div} \tau} \qquad (3)$$

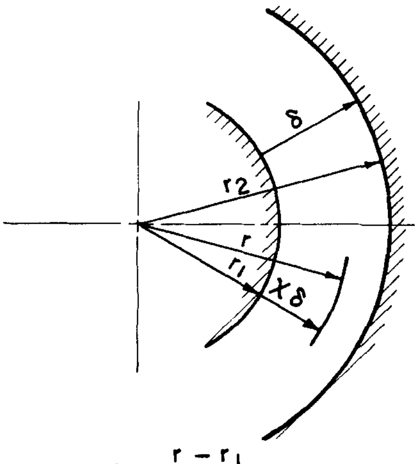
The left-hand terms in Equations (3) represent the mass times acceleration of a fluid element, term (a) being the gradient of the translational kinetic energy of the fluid and term (b) arising from the vorticity of the flow. Terms (*c*) represent the forces due to the pressure and external force fields acting on the fluid, and term (d) represents the viscous forces.

It is suggested that when the magnitude of the acceleration force (b) reaches a certain multiple of the magnitude of the viscous force (d), the fluid motion will be unstable to certain types of disturbances and stable laminar flow will no longer exist. Mathematically this suggestion may be expressed as

$$|\rho \mathbf{v} \times \zeta| = K\, |\operatorname{div} \tau| \qquad (4)$$

At this point some rather general properties of the stability parameter K may be pointed out. From the nature of its definition K is a local parameter and therefore a function of position in the flow field. It is inherently a positive number. The term $\mathbf{v} \times \zeta$ vanishes at all solid boundaries and along the lines of symmetry of the velocity profile, whereas div τ does not. Therefore it follows that K must also vanish on solid boundaries and along the lines of symmetry of the velocity field. Hence $K \geqq 0$ everywhere, and at some point in the flow region K acquires a maximum value which shall be designated by the symbol $\overline{K}$.

It will be shown below, for Newtonian flows, that $\overline{K}$ is proportional to



$$X = \frac{r - r_1}{r_2 - r_1}$$

$$0 \leq X \leq 1$$

Fig. 1. Coordinate system for annuli.

the bulk flow Reynolds number. When the Reynolds number of a laminar flow has increased to a sufficiently large value, transition to turbulence occurs. It is postulated that when $\bar{K}$ reaches a sufficiently large constant critical value κ at some point in the flow field, certain types of disturbances, if introduced at that point, will be able to grow and spread to the solid boundary surfaces. Here they can become self-maintaining and give rise to a general transition to turbulence throughout the region of flow.

Equation (4) may be considerably simplified for the special case of steady state rectilinear flows for which one notes

$$\mathbf{v} \times \zeta = \frac{1}{2}\,\mathbf{grad}\,(\mathbf{v}\cdot\mathbf{v}) \qquad (5)$$

$$\mathrm{div}\,\tau = \mathbf{F} - \mathbf{grad}\,p \qquad (6)$$

Using Equations (5) and (6) one may write Equation (4) as

$$\frac{1}{2}\rho\frac{|\mathbf{grad}\,(\mathbf{v}\cdot\mathbf{v})|}{|\mathbf{F} - \mathbf{grad}\,p|} = K \qquad (7)$$

Before Equation (7) is applied to certain rectilinear Newtonian flows, it will be convenient to introduce a special Reynolds number.

Lohrenz and Kurata (5) obtained an expression for an equivalent diameter which, if used in the conventional definitions of the friction factor and Reynolds number, permits all laminar flow data for pipes, concentric annuli, and parallel plates to be represented by the relationship

$$f = 16\,N_{Re}^{-1} \qquad (8)$$

where $N_{Re} = d_e\bar{v}\rho/\mu$ and $f = d_e(-dp/dz)/(2\rho\bar{v}^2)$.

The equivalent diameter may be expressed as

$$d_e = \delta\sqrt{8\,\Psi(\epsilon)} \qquad (9)$$

where $\delta = r_2 - r_1$ (see Figure 1), and

$$\Psi(\epsilon) = \frac{[1 + (1+\epsilon)^2]\ln(1+\epsilon) + 1 - (1+\epsilon)^2}{2\epsilon^2\ln(1+\epsilon)} \qquad (10)$$

By repeated applications of L'Hôpital's rule one may show

$$\lim_{\epsilon\to 0}\Psi(\epsilon) = \frac{1}{3} \qquad (11)$$

(Parallel Plates)

$$\lim_{\epsilon\to\infty}\Psi(\epsilon) = \frac{1}{2} \qquad (12)$$

(Pipes)

PIPE FLOW

For the case of flow in an open pipe Equations (9) and (12) show that $d_e = 2\delta = D$ and hence $N_{Re} = Dv\rho/\mu$, the conventional pipe flow Reynolds number.

Equation (7) may be written in general for this geometry as

$$K = \frac{1}{2}\rho\frac{d(v^2)/dr}{dp/dz} \qquad (13)$$

Upon introduction of the Poiseuille velocity profile expression Equation (13) is reduced to

$$K = \frac{1}{2}N_{Re}\,\xi(1-\xi^2) \qquad (14)$$

where $\xi = r/r_w$. The value $\bar{\xi}$ (the point where $K = \bar{K}$) is found by differentiating Equation (14) with respect to ξ, equating the resultant expression to zero, and solving for $\bar{\xi}$. The result of this operation is the same as that found by Ryan and Johnson (*1*) and is in agreement with the experimental observations of Gibson (*6*) and Leite and Kuethe (*7*); namely

$$\bar{\xi} = \sqrt{1/3} \qquad (15)$$

By introducing Equation (15) into Equation (14) one finds

$$\bar{K} = \sqrt{\frac{1}{27}}N_{Re} \qquad (16)$$

If in Equation (16) one puts $N_{Re} = N_{Re_c} = 2{,}100$ and $\bar{K} = \kappa$, one obtains

$$\kappa = 404 \qquad (17)$$

Comparison of Equations (1) and (16) shows that $2\kappa = Z$, and hence Equation (1) is contained as a special case of Equation (7).

FLOW BETWEEN PARALLEL PLATES

For flow between parallel plates Equations (9) and (11) give $d_e = h\sqrt{32/3}$, where $h = \delta/2$ is the half separation of the plates.

Equation (7) becomes $K = \frac{1}{2}\frac{\rho}{dp/dz}d(v^2)/dy$. Following the same procedure as above, but using $v(y)$ for plane Poiseuille flow, one is led to the following result:

$$K = \frac{3}{8}\sqrt{\frac{3}{2}}N_{Re}\,\Phi(1-\Phi^2) \qquad (18)$$

where $\Phi = y/h$. If Equation (18) is differentiated with respect to Φ, equated to zero, and solved for $\bar{\Phi}$ (the value of Φ for which $K = \bar{K}$), one finds

$$\bar{\Phi} = \sqrt{1/3} \qquad (19)$$

If one introduces Equation (19) into Equation (18), solves for N_{Re}, and replaces $\bar{K}$ by κ from Equation (17), one finds

$$N_{Re_c}(\text{Parallel Plates}) = 2{,}288 \qquad (20)$$

This result may be compared with the experimental observations of Davies and White (*3*) from whose data one can compute $N_{Re_c} = 2{,}285$. Thus for Newtonian flow between parallel plates the generalized parameter defined by Equation (4) may be used to predict a critical Reynolds number which is in excellent agreement with experimental observations. It should be noticed at this point that whereas the magnitude of the critical Reynolds number for flow between parallel plates is different from that for pipe flow, the *same* value of κ applies in both cases.

FLOW IN CONCENTRIC ANNULI

Figure 1 is a diagram showing a convenient coordinate system to use in

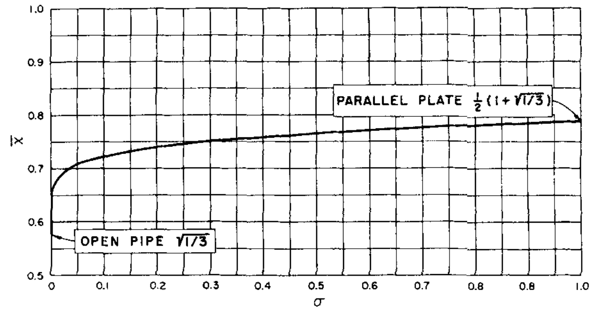


Fig. 2. Variation of $\bar{\chi}$ with σ.

the treatment of flow in concentric annuli. The solution to the equations of motion for the rectilinear flow of a Newtonian fluid in an annulus may be expressed in these coordinates as

$$v(\chi) = \frac{1}{4}\frac{\delta^2}{\mu}\left(-\frac{dp}{dz}\right)\left\{\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}\ln(1+\epsilon\chi) + 1-(1+\epsilon\chi)^2\right\}\frac{1}{\epsilon^2} \qquad (21)$$

In terms of the variable χ Equation (7) becomes

$$K = \frac{1}{2}\frac{\rho}{\delta}\frac{d(v^2)/d\chi}{dp/dz} \qquad (22)$$

By the use of Equation (21) and the definition of the area mean velocity, Equation (22) may be written as

$$K = -\frac{N_{Re}}{8\sqrt{2\Psi(\epsilon)}} \frac{\left[\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}\ln(1+\epsilon\chi)+1-(1+\epsilon\chi)^2\right]\left[\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}-2(1+\epsilon\chi)^2\right]}{\epsilon^3\Psi(\epsilon)\,(1+\epsilon\chi)} \qquad (23)$$

where $\Psi(\epsilon)$ is defined by Equation (10).

As a check of Equation (23) one may apply L'Hôpital's rule to obtain as a limit for parallel plates

$$\lim_{\epsilon\to 0} K = -\frac{3}{2}\sqrt{\frac{3}{2}}\,N_{Re}\chi(1-\chi)(1-2\chi) \qquad (24)$$

In this limit $\chi = (1+\Phi)/2$, and Equation (24) is equivalent to Equation (18).

The limit of Equation (23) for pipe flow is

$$\lim_{\epsilon\to\infty} K = \frac{1}{2}N_{Re}\chi(1-\chi^2) \qquad (25)$$

Since in this limit $\chi = \xi$, Equation (25) is equivalent to Equation (14).

For intermediate values of ϵ the coordinate $\bar{\chi}$, at which $K = \bar{K}$, is found by equating $dK/d\chi$ to zero and solving for the root. The equation to be solved is

$$0 = \frac{1}{\epsilon^2(1+\epsilon\bar{\chi})^2}\left[\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}-2(1+\epsilon\bar{\chi})^2\right]^2 - \frac{1}{\epsilon^2(1+\epsilon\bar{\chi})^2}\left[\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}+2(1+\epsilon\bar{\chi})^2\right]\left[\frac{\epsilon(2+\epsilon)}{\ln(1+\epsilon)}\ln(1+\epsilon\bar{\chi})+1-(1+\epsilon\bar{\chi})^2\right] \qquad (26)$$

The open tube limit for $\bar{\chi}$ is given by Equation (15). The corresponding limit for parallel plates is $\bar{\chi} = \frac{1}{2}(1+\bar{\Phi})$, where $\bar{\Phi}$ is given by Equation (19). These limits are shown in Figure 2 which is a plot of the values of $\bar{\chi}$, obtained by numerical solution of Equation (26), as ordinate with $r_1/r_2 = \sigma$ as abscissa.

One may use Equations (17), (23), and the values of $\bar{\chi}$ from Figure 2 to compute N_{Re_c} (annuli) as a function of σ. The solid curve in Figure 3 was obtained in this manner. Both Figures 2 and 3 reveal the striking effect of a tiny central core. Figure 3 in particular shows an interesting result of the present treatment. The theoretical equations predict a maximum critical Reynolds number equal to 2,462 at $\sigma \simeq 0.15$.

COMPARISON WITH EXPERIMENT

The literature was searched for data with which to test the present predictions. Much of the available data, especially for small σ values, are inadequate in the laminar region, so comparisons with theory cannot be made. Other data indicate that the flows were not rectilinear, thus invalidating the data for comparison with the present calculations.

The data which were acceptable seemed to represent three distinct types of flow behavior. Some flows, particularly for open pipes, parallel plates, and narrow (large σ) annuli, exhibited the required characteristics of rectilinearity. This condition is easily tested from the data by a linear plot of the product of f and N_{Re} as ordinate with N_{Re} as abscissa. Truly rectilinear flow is characterized by a unique constant value of the $f\,N_{Re}$ product for the entire laminar region. The laminar-turbulent transition is marked by the departure of the data from this constant value along some well-defined curve. All of the solid data points in Figure 3 correspond to flows of this type.

A second type of flow characteristic (often associated with annuli of lower σ values) was the existence of two distinct constant values of the product $f\,N_{Re}$ in the laminar range. The data seemed to be well represented by the theoretical value of $f\,N_{Re}$ for low Reynolds numbers. A somewhat higher but constant value was required for intermediate Reynolds numbers up to a well-defined point of transition to a type of flow which was clearly nonlaminar. Several sets of the rather extensive data of Walker and coworkers (8) indicated such flow characteristics. This behavior was not indicated by their open pipe laminar flow data.

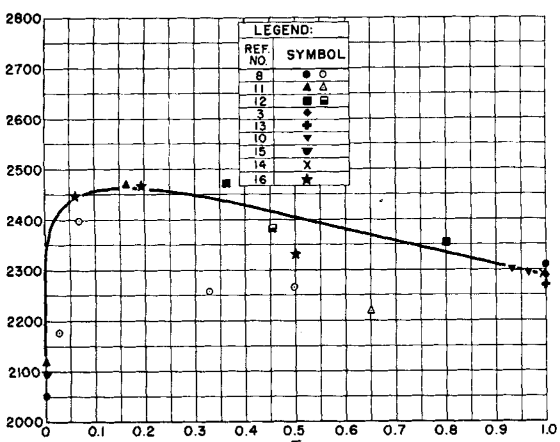


Fig. 3. Variation of critical Reynolds number with σ.

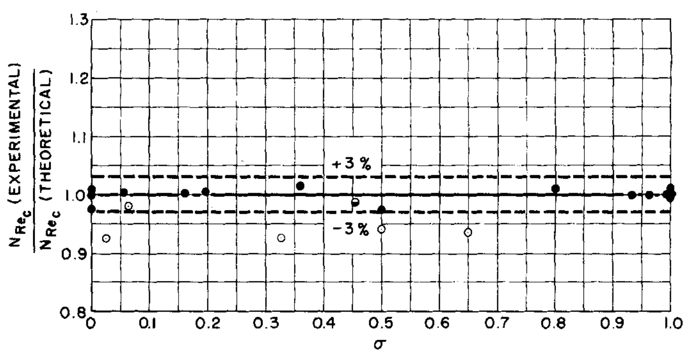


Fig. 4. Comparison of experimental and theoretical critical Reynolds number.

Their data of this second type were included as a matter of interest, since they showed the predicted trend of N_{Re_c} with σ and covered the important range of low σ. Data corresponding to this second type of flow characteristics are shown in Figure 3 as open points.

The half-shaded point in Figure 3 represents a set of data for which the flow was apparently rectilinear in the laminar range, but for which insufficient data were given in the turbulent-transition range to enable one to make an accurate estimate of the critical Reynolds number. The plotted value represents a N_{Re} below which the flow is evidently laminar.

A third type of flow behavior suggested by some of the literature data (*8*, *9*) was the complete absence of a unique constant for the fN_{Re} product at all values of N_{Re}, thus making it impossible to assign any definite transition Reynolds number. When these data were plotted as described, a smooth curve could be drawn through the points from the lowest to the highest Reynolds numbers with no indication of a distinct change in the type of flow. Such data were not considered further.

DISCUSSION OF RESULTS

From Figure 3 it is evident that those data which represented rectilinear laminar flows (solid points) are in good agreement with the present calculations and clearly confirm the reality of the maximum in the curve. The data for flows of the second type which do not exactly satisfy the laminar flow criterion (open points in Figure 3) are not inconsistent with the calculated results. The data of reference *8* clearly show the marked increase in N_{Re_c} with increasing σ, and (with the exception of the point for $\sigma = 0.3312$) also show the general qualitative behavior as suggested by the calculated curve. Since these data do not comply fully with the laminar flow requirements, their agreement or lack thereof must not be given as much weight as for the solid points. However it is interesting to observe the predicted trends, even in these data.

Figure 4 is a plot showing the ratio of the experimental critical Reynolds number divided by the theoretical value as ordinate with σ as abscissa. The same convention of shading has been used as in Figure 3. It will be observed that all of the solid points and the half-shaded point lie within $\pm 3\%$ of the value unity.

SUMMARY

A generalized stability parameter has been suggested which is based upon a physical analogy with the familiar Reynolds number and a consideration of pertinent terms in the equations of motion. This parameter was applied to the case of rectilinear flow of Newtonian fluids in pipes, concentric annuli, and parallel plates. The equations for the annular flow system reduce to the equations for pipe flow and plane Poiseuille flow as limiting cases. These equations predict a maximum in the dependence of critical Reynolds numbers on σ which can be verified quantitatively from existing literature data, although further investigation of the range $0 \leqq \sigma \leqq 0.2$ would seem desirable.

The results of the present paper together with reference 5 provide a unified method for the treatment of laminar Newtonian flow data and the transition from laminar to turbulent flow in pipes, concentric annuli, and between parallel plates.

Although the parameter is applied here only to Newtonian flows, since it contains Ryan and Johnson's parameter for pipe flow as a special case, their experimental verification (*1*) of it for isothermal flow of power-law non-Newtonian fluids and Hanks and Christiansen's (*2*) experimental verification of it for heated flows of similar fluids are valid. Hence the new parameter is seen to have quite wide application. The utility of this generalized parameter is that it circumvents the problem of defining a viscosity for non-Newtonian fluids. Furthermore its critical value is a constant independent of the geometry of the flow field, the shape of the velocity profile (*2*), or the existence of temperature gradients in the flow field (*2*).

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NOTATION

d_e = equivalent diameter defined by Equation (9)
D = tube diameter
f = friction factor, $d_e(-dp/dz)/(2\rho\bar{v}^2)$
h = half separation between parallel plates
K = new local stability parameter defined by Equation (4)
$\overline{K}$ = maximum value of K in cross section of flow
N_{Re} = Reynolds number, $d_e\bar{v}\rho/\mu$
$-dp/dz$ = axial pressure gradient
r = radial coordinate
r_1 = radius of inner cylinder in concentric annulus
r_2 = radius of outer cylinder in concentric annulus
r_w = radius of pipe
v = magnitude of velocity vector
$\bar{v}$ = area mean velocity (volume flow divided by cross-sectional area)
y = distance measured from the center line between parallel plates
Z = Ryan and Johnson's stability parameter

Greek Letters

δ = separation between walls of concentric annulus
ϵ = δ/r_1
κ = critical value of stability parameter = 404
μ = viscosity of Newtonian fluid
ξ = dimensionless pipe flow radial coordinate, r/r_w
ρ = density of fluid
σ = radius ratio for annuli, r_1/r_2
Φ = dimensionless co-ordinate for flow between parallel plates, y/h
χ = dimensionless co-ordinate for annular flow, $(r-r_1)/(r_2-r_1)$
$\Psi(\epsilon)$ = function defined by Equation (10)

Vector and Tensor Quantities

$\mathbf{F}$ = external body force acting on fluid element
ζ = vorticity, $\zeta = \mathbf{curl\ v}$
$\mathbf{v}$ = velocity
τ = stress deviation tensor

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